Received: *April 14, 2000*, Accepted: *May 17, 2000*

Renormalization group flows on D3 branes at an orbifolded conifold

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Abstract: We consider D3-branes at an orbifolded conifold whose horizon X_5 resolves into a smooth Einstein manifold which joins several copies of $\mathbb{T}^{1,1}$. We describe in details the resolution of the singular horizon X_5 and describe different types of two-cycles appearing in the resolution. For a large number of D3 branes, the AdS/CFT conjecture becomes a duality between type IIB string theory on $AdS_5 \times X_5$ and the $\mathcal{N} = 1$ field theory living on the D3 branes. We study the fractional branes as small perturbations of the string background and we reproduce the logarithmic flow of field theory couplings by studying fluxes of NS-NS and R-R two forms through different 2-cycles of the resolved horizon.

Keywords: D-branes, String Duality.

Contents

1. Introduction

In the last years we have seen increasing evidences that string/M theory on AdS spaces are dual to large-N strongly coupled gauge theories [10, 11]. The most extensively studied cases are the dualities between type-IIB string theories on $AdS_5 \times M_5$ for positively curved five dimensional Einstein manifolds and large-N strongly coupled four dimensional conformal gauge theories. The simplest example is $AdS_5 \times S^5$. In this case, the dual field theory is $\mathcal{N} = 4$ supersymmetric gauge theory. In [12, 13], field theories with less supersymmetry have been studied as duals to string theory on orbifolds of S^5. In [18], Klebanov and Witten studied the Einstein manifold $T^{1,1} = (\mathrm{SU}(2) \times \mathrm{SU}(2))/\,\mathrm{U}(1)$. This was the first example of five dimensional spaces which are not orbifolds of S^5. type-IIB string theory compactified on this manifold is dual to an $\mathcal{N} = 1$ superconformal $\mathrm{SU}(N) \times \mathrm{SU}(N)$ gauge theory with a quartic superpotential for bifundamental fields. In the T-dual picture, D3 branes probing a metric cone over $T^{1,1}$ (which is the conifold) is either a brane configuration with D4 branes together with orthogonal NS branes [7, 8] or a brane box with D5 branes together with orthogonal NS branes [14, 15]. Other results on the conifold or the quotients of the conifold and their field theory duals were obtained in [1, 20]–[31].

Another exciting development was study of the gravity dual of the field theory Renormalization group flow. The main point is that the radial coordinate of AdS_5 has the natural interpretation as an energy scale in field theory. Thus it becomes natural to consider type-IIB supergravity interpolating solutions where the metric and the fields depend on the radial coordinate, and to interpret these solutions as RG flows in the dual field theory. Many ideas have emerged concerning different aspects of supersymmetric and non-supersymmetric RG flows for four dimensional theories [32]–[38, 48, 40].

Because the AdS/CFT involves the full string theory we should be able to go beyond the Supergravity approximation. In [1] Klebanov and Nekrasov have studied the gravity duals of fractional branes in supersymmetric conifold and orbifold theories where the presence of the fractional branes breaks the conformal invariance and introduces an RG flow in field theories. One of the models considered in [1] involves a large number of D3-branes at a conifold singularity whose near-horizon is a $AdS_5 \times \mathbb{T}^{1,1}$ background and the fluxes of B^{NS} and B^{RR} forms through the blow-up 2-cycle determine a difference in the coupling constants of the two group factors appearing on the world-volume of the D3 branes at singularities. The supergravity equations of motion together with the specific formulas for the 2-forms and 3-forms on $\mathbb{T}^{1,1}$ give a solution which reproduces the logarithmic flow of field theory beta function. Previous results were obtained in type-0B string where effective action uncertainties occur [2]–[6]. In all the studies of RG flow from AdS/CFT, the RG flow was determined by turning on different operators in the field theory which break the conformal invariance. In supergravity this means turning on some of the supergravity scalar fields. Another way to break the conformal invariance is to introduce the twisted sectors of string theories.

In this paper we go one step further and study D3 branes on an orbifolded conifold which is the quotient of the conifold by $\mathbb{Z}_k \times \mathbb{Z}_l$. Now the horizon X_5 will be singular along two disjoint, but linked circles and we need to resolve the singularities in order to obtain a smooth Einstein manifold $\widetilde{X_5}$. We completely describe the resolution of the orbifolded conifold itself in two steps and discover that there are kl isolated conifold singularities after the first step of the resolution. After the resolution we find a smooth Einstein manifold $\widetilde{X_5}$ containing $kl+k+k-2$ two-cycles. As $\widetilde{X_5}$ approaches the exceptional fiber of the first resolution, kl cycles are vanishing into the singular points, and $k+l-2$ cycles deform to cycles in the fiber which separate the two circles of singularities. Near each singular point, $\widetilde{X_5}$ can be approximated by $\mathbb{T}^{1,1}$ and kl two-cycles of $\widetilde{X_5}$ come from these $\mathbb{T}^{1,1}$'s. We then consider a large number of D3 branes probing this singularity, which corresponds to a brane box with D5 branes and orthogonal NS branes via T-duality [14, 15, 24]. The D5 branes wrapped on 2-cycles of $\widetilde{X_5}$ vanishing into the singular points of the partially resolved orbifolded conifold are the fractional D3 branes. We study the B^{NS} and B^{RR} fluxes through different 2-cycles of $\widetilde{X_5}$ which give rise to a logarithmic flow for the field theory coupling constants. This agrees with the field theory expectations for the RG flow.

In section 2 we study the geometry of the orbifolded conifold and describe how to obtain a smooth horizon from the singular horizon $\mathbb{T}^{1,1}/\mathbb{Z}_k \times \mathbb{Z}_l$. We also identify the different fractional D3 branes in the singularity picture with different components of brane interval or brane box configurations obtained by T-dualities. In section 3 we describe the supergravity dual to the field theory Renormalization Group flows.

2. Geometry and brane configurations of orbifolded conifolds

In this section we study the geometry of the orbifolded conifolds $\mathcal{C}_{kl}$ in detail and we make connections with brane configurations obtained by T-dualities. In particular, we study the resolutions of the orbifolded conifolds $\mathcal{C}_{kl}$ and the associated fractional

JHEP05(2000)030

branes in the T-dual picture. At the end of the section, we describe the homological cycles of the resolved horizon of $\mathcal{C}_{kl}$, and thus extends the results of [19, 18]. This play an important role in the study of fluxes in the next section.

Consider a singular Calabi-Yau threefold Y_6 which is a metric cone over a five dimensional Einstein manifold M_5. Then the metric near the apex of the cone will be

$$ds^2_{Y_6} = dr^2 + r^2 ds^2_{M_5} \,. \tag{2.1}$$

Here the apex is located at $r = 0$ and M_5 is called the horizon of the cone Y_6. If N parallel D3 branes are placed at the apex of the cone Y_6, the resulting ten dimensional spacetime has the metric

$$ds^2 = R^2 \left[\frac{r^2}{R^4}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + \frac{dr^2}{r^2} + ds_5^2 \right], \qquad R^4 \sim g_s N(\alpha')^2 \,. \tag{2.2}$$

The near-horizon ($r \to 0$) limit of the geometry is $AdS_5 \times M_5$. Type-IIB theory on this background is conjectured to be dual to the conformal limit of the field theory on the D3 branes.

In [18], an example of such duality has been discussed in the case of a conifold whose horizon is $\mathbb{T}^{1,1} = (\mathrm{SU}(2) \times \mathrm{SU}(2))/\,\mathrm{U}(1)$. The conifold is a three dimensional hypersurface singularity in $\mathbb{C}^4$ defined by:

$$\mathcal{C}: \quad z_1 z_2 - z_3 z_4 = 0 \,. \tag{2.3}$$

The conifold can be realized as a holomorphic quotient of $\mathbb{C}^4$ by the $\mathbb{C}^*$ action given by [18]

$$(A_1, A_2, B_1, B_2) \mapsto (\lambda A_1, \lambda A_2, \lambda^{-1} B_1, \lambda^{-1} B_2) \quad \text{for } \lambda \in \mathbb{C}^* \,. \tag{2.4}$$

Thus the charge matrix is the transpose of $Q' = (1, 1, -1, -1)$ and $\Delta = \sigma$ will be a convex polyhedral cone in $\mathbf{N}'_{\mathbb{R}} = \mathbb{R}^3$ generated by $v_1, v_2, v_3, v_4 \in \mathbf{N}' = \mathbb{Z}^3$ where

$$v_1 = (1,0,0)\,, \qquad v_2 = (0,1,0)\,, \qquad v_3 = (0,0,1)\,, \qquad v_4 = (1,1,-1)\,. \tag{2.5}$$

The isomorphism between the conifold $\mathcal{C}$ and the holomorphic quotient is given by

$$z_1 = A_1 B_1\,, \qquad z_2 = A_2 B_2\,, \qquad z_3 = A_1 B_2\,, \qquad z_4 = A_2 B_1\,. \tag{2.6}$$

To identify the horizon from this point of view, note that we can divide by the scaling $z_i \to s z_i$ (with real positive s) by setting $|A_1|^2 + |A_2|^2 = |B_1|^2 + |B_2|^2 = 1$. This gives us $S^3 \times S^3 = \mathrm{SU}(2) \times \mathrm{SU}(2)$. Then dividing by the $\mathrm{U}(1)$ action

$$(A_1, A_2, B_1, B_2) \mapsto (e^{i\alpha} A_1, e^{i\alpha} A_2, e^{-i\alpha} B_1, e^{-i\alpha} B_2), \tag{2.7}$$

we obtain $\mathbb{T}^{1,1} = (\mathrm{SU}(2) \times \mathrm{SU}(2))/\,\mathrm{U}(1)$.

The five dimensional manifold $\mathbb{T}^{1,1}$ has 2-cycles and 3-cycles. Besides the D3 branes orthogonal to $\mathbb{T}^{1,1}$, there are wrapped D3 branes over the 3-cycles of $\mathbb{T}^{1,1}$

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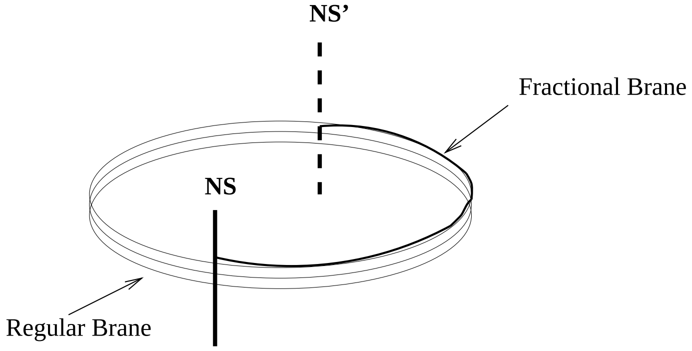

Figure 1: An interval model.

(which correspond to "dibaryon" operators [27]) and wrapped D5 branes over 2-cycles of $\mathbb{T}^{1,1}$ (which to correspond to domain walls in AdS_5 [27] and to fractional D3 branes [16, 17, 9]). Because $\mathbb{T}^{1,1}$ is $S^2 \times S^3$, we can identify the 2-cycle with S^2 and the 3-cycle with S^3. These two cycles are orthogonal so the D3 brane wrapped on S^3 is orthogonal to the D5 brane wrapped on S^2, therefore when they cross each other a fundamental string is created as explained in [43, 44, 45] and the gauge group becomes $\mathrm{SU}(N+1) \times \mathrm{SU}(N)$.

The geometrical picture is T-dual to different types of brane configuration. By one T-duality one can obtain the brane interval picture with D4 branes wrapped on a circle and by two T-dualities one obtains the brane box picture with D5 branes wrapped on a 2-torus. The fractional branes have also been identified in the brane interval picture in [9]. The idea was to interpret the conifold (2.3) as a $\mathbb{C}^*$ fibration over the $\mathbb{C}^2$ parameterized by z_3, z_4. By performing T-duality along the U(1)-orbit in the $\mathbb{C}^*$-fiber, we obtain from the degenerate fibers $z_1 = 0$ and $z_2 = 0$, two NS fivebranes extended, say, $x^0x^1x^2x^3x^4x^5$ and $x^0x^1x^2x^3x^8x^9$ directions which we denote by NS and NS' branes. The D3 branes located at the singular point transform into D4 branes wrapping a circle which is transverse to the NS fivebranes. $\mathbb{T}^{1,1}$ has a U(1)-fibration over $\mathbb{P}^1 \times \mathbb{P}^1$ and a two cycle S^2 of $\mathbb{T}^{1,1}$ can be identified to the difference of two homologically distinct spheres coming from $\mathbb{P}^1 \times \mathbb{P}^1$. After identifying $\mathbb{P}^1 \times \mathbb{P}^1$ with the exceptional locus in the full resolution of the conifold, D5 brane wrapping the two cycle S^2 will transform as a D4 brane wrapping on one interval between two NS-branes. This is a fractional brane in the interval model (figure 1).

One of the goals of this paper is to study the fractional branes in the brane box model for a quotient of the conifold. To do this, we start by taking a further quotient of the conifold $\mathcal{C}$ by a discrete group $\mathbb{Z}_k \times \mathbb{Z}_l$. Here $\mathbb{Z}_k$ acts on A_i, B_j by

$$(A_1, A_2, B_1, B_2) \mapsto (e^{-2\pi i/k} A_1, A_2, e^{2\pi i/k} B_1, B_2)\,, \tag{2.8}$$

and $\mathbb{Z}_l$ acts by

$$(A_1, A_2, B_1, B_2) \mapsto (e^{-2\pi i/l} A_1, A_2, B_1, e^{2\pi i/l} B_2)\,. \tag{2.9}$$

Thus they will act on the conifold $\mathcal{C}$ by

$$(z_1, z_2, z_3, z_4) \mapsto (z_1, z_2, e^{-2\pi i/k} z_3, e^{2\pi i/k} z_4) \tag{2.10}$$

and

$$(z_1, z_2, z_3, z_4) \mapsto (e^{-2\pi i/l} z_1, e^{2\pi i/l} z_2, z_3, z_4)\,. \tag{2.11}$$

Its quotient is called the orbifolded conifold (or the hyper-quotient of the conifold) and denoted by $\mathcal{C}_{kl}$.

Note that the action (2.8) leaves a complex two space $A_1 = B_1 = 0$ fixed in $\mathbb{C}^4$ and this is isomorphic to $\mathbb{C}^1$ given by $z_1 = z_3 = z_4 = 0$ on the conifold $\mathcal{C}$ after dividing by the U(1) action. Similarly, the action (2.9) leaves fixed a complex two space $A_1 = B_2 = 0$ in $\mathbb{C}^4$ and it is isomorphic to $\mathbb{C}^1$ given by $z_1 = z_2 = z_3 = 0$ on the conifold $\mathcal{C}$ after dividing by the U(1) action. Furthermore, the action (2.8) descends to the horizon $\mathbb{T}^{1,1}$ and leaves the following circle fixed:

$$|z_2|^2 = 1\,, \qquad z_1 = z_3 = z_4 = 0 \tag{2.12}$$

or equivalently $|A_2|^2 = |B_2|^2 = 1, A_1 = B_1 = 0 \pmod{\mathrm{U}(1)}$. Similarly, the action (2.9) leaves the following circle fixed

$$|z_4|^2 = 1, \quad z_1 = z_2 = z_3 = 0 \tag{2.13}$$

or equivalently $|A_2|^2 = |B_1|^2 = 1, A_1 = B_2 = 0 \pmod{\mathrm{U}(1)}$. Hence the horizon $X_5 := \mathbb{T}^{1,1}/\mathbb{Z}_k \times \mathbb{Z}_l$ of the orbifolded conifold is singular along these two circles. These two circles are separated but linked. The horizon X_5 has $\mathbf{A}_{k-1}$ singularity along the circle (2.12) and $\mathbf{A}_{l-1}$ singularity along the circle (2.13). String theory in the back ground $AdS_5 \times X_5$ has massless fields which are localized along these two linked circles. As discussed in [15], these massless fields are the twisted modes and they propagate on $AdS_5 \times S^1 \sqcup AdS_5 \times S^1$ where $S^1 \sqcup S^1$ are the circles of singularities. As we shall see below, there are $k + l - 2$ 2-cycles which separate these two circles. The fluxes of the NSNS and RR two forms through these cycles give rise to scalars which live in the $AdS_5 \times S^1$ space and are the same as the scalars introduced in section 3 of [1].

To put the actions (2.6), (2.8) and (2.9) on an equal footing, consider the over-lattice $\mathbf{N} = \mathbf{N}' + \frac{1}{k}(v_3 - v_1) + \frac{1}{l}(v_4 - v_1)$. Now the lattice points $\sigma \cap \mathbf{N}$ of σ in $\mathbf{N}$ are generated by $(k+1)(l+1)$ lattice points as a semigroup. The discrete group $\mathbb{Z}_k \times \mathbb{Z}_l \cong \mathbf{N}/\mathbf{N}'$ will act on the conifold $\mathbb{C}^4//\,\mathrm{U}(1)$ and its quotient will be the symplectic reduction $\mathbb{C}^{(k+1)(l+1)}//\,\mathrm{U}(1)^{(k+1)(l+1)-3}$. The new toric diagram for $\mathcal{C}_{kl}$ will also lie on a plane at a distance from the origin and the toric diagram on the plane for $\mathcal{C}_{23}$ is shown in figure 1.

In suitable coordinates, the orbifolded conifold will be given by

$$\mathcal{C}_{kl} : xy = z^l\,, \qquad uv = z^k\,. \tag{2.14}$$

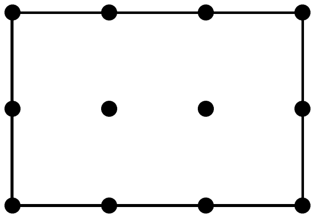

Figure 2: A toric diagram for $\mathbb{Z}_2 \times \mathbb{Z}_3$ hyper-quotient of the conifold, $\mathcal{C}_{23}$.

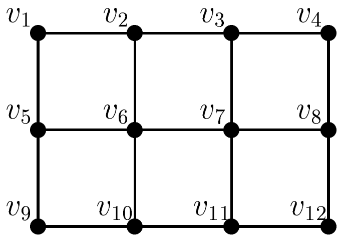

Figure 3: A toric diagram for $\widetilde{\mathcal{C}_{23}}$.

As we have seen above, the horizon X_5 is singular. To obtain a smooth Einstein manifold from X_5, we will resolve the singularities of $\mathcal{C}_{kl}$ itself. We resolve the singular threefold $\mathcal{C}_{kl}$ in two steps. In the first step, we choose a partial resolution, denoted by $\widetilde{\mathcal{C}_{kl}}$, of the orbifolded conifold $\mathcal{C}_{kl}$ for which the horizon will be smooth, but the Calabi-Yau threefold $\widetilde{\mathcal{C}_{kl}}$ will have kl number of isolated singular points. Around each singular point, the Calabi-Yau space $\widetilde{\mathcal{C}_{kl}}$ is locally a metric cone over an Einstein manifold $\mathbb{T}^{1,1}$. In terms of the toric diagram, the partial resolution we have chosen is obtained by adding all possible vertical and horizontal arrows to the toric diagram of $\mathcal{C}_{kl}$. For example, the toric diagram for $\widetilde{\mathcal{C}_{23}}$ is given as in figure 3.

We are going to describe in detail each step but let us discuss first some features and make the connection of the result with the T-dual brane configurations. The partially resolved space $\widetilde{\mathcal{C}_{kl}}$ is covered by kl squares and each square in the toric diagram represents an ordinary conifold. Thus the metric near each singular point can be written locally as follows

$$ds^2 = \frac{1}{9}(d\phi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2 + \frac{1}{6}\sum_{a=1}^{2}(d\theta_a^2 + \sin^2\theta_a d\phi_a^2)\,. \qquad (2.15)$$

Note that $\mathcal{C}_{kl}$ can be regarded as a $\mathbb{C}^* \times \mathbb{C}^*$ fibration over the z-plane via (2.14). By taking T-duality along $\mathrm{U}(1) \times \mathrm{U}(1)$ orbit in $\mathbb{C}^* \times \mathbb{C}^*$, we will have two types of NS branes extended in, say, $x^0x^1x^2x^3x^4x^5$ and $x^0x^1x^2x^3x^8x^9$ directions, where x^4, x^8 are compact directions coming from the degenerate $\mathrm{U}(1) \times \mathrm{U}(1)$-orbit. The separation of the NS branes along the x^8 direction and similarly that of the NS' branes along the x^4 direction can be achieved by partially resolving $\mathcal{C}_{kl}$. But there will be kl intersections of the NS and NS' branes on the x^4x^8 torus which correspond to the singular points of the partially resolved orbifolded conifold $\widetilde{\mathcal{C}_{kl}}$. By replacing these singular points by $\mathbb{P}^1$'s, we resolve the singularities of $\widetilde{\mathcal{C}_{kl}}$. In the field theory, the process of blowing-up and turning the NS-NS B-fluxes through the kl $\mathbb{P}^1$ cycles means turning on gauge couplings. In brane box configurations it means replacing the intersection of NS and NS' branes with 'diamonds' [24], the size of the diamonds being given by the fluxes of $NSNS$ fields through the blow-up cycles. In this T-duality, D3 branes become D5 branes which fill the x^4x^8 directions and the resulting brane configuration is as a brane box model shown in figure 4 [15, 24]. Its components are diamonds and boxes

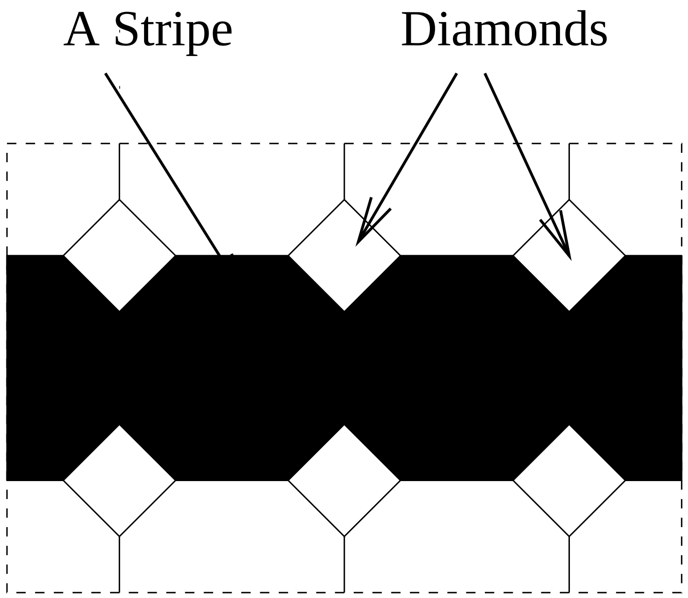

Figure 4: A brane box model.

and a stripe is an horizontal or vertical line of boxes (in figure 4 we have represented an horizontal stripe).

In the second step we completely resolve the singularities of $\widetilde{\mathcal{C}_{kl}}$ by replacing each of the singular points by a copy of $\mathbb{P}^1$ as explained before, procedure called a small resolution. In terms of the toric diagram, this corresponds to joining a pair of the diagonal vertices (but not both pairs) by a line segment in each square. Let us denote this completely resolved threefold by $\widehat{\mathcal{C}_{kl}}$. For example, the toric diagram for $\widehat{\mathcal{C}_{23}}$ is given as in figure 3.

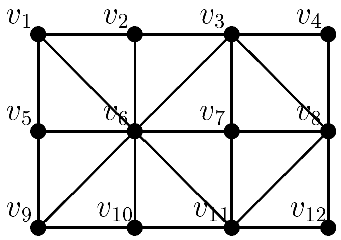

Figure 5: A toric diagram for $\widehat{\mathcal{C}_{23}}$.

Let

$$\widetilde{\pi} : \widetilde{\mathcal{C}_{kl}} \to \mathcal{C}_{kl}, \quad \widehat{\pi} : \widehat{\mathcal{C}_{kl}} \to \widetilde{\mathcal{C}_{kl}} \tag{2.16}$$

be the first and the second resolution discussed above. Let $\mathbf{o}$ be the apex of the cone $\mathcal{C}_{kl}$ and

$$\widetilde{X_5} = \widetilde{\pi}^{-1}(X_5)\,. \tag{2.17}$$

Note that $\widetilde{\mathcal{C}_{kl}}$ is covered by a kl number of the ordinary conifolds corresponding to the squares in the toric diagram. Hence $\widetilde{\mathcal{C}_{kl}}$ has kl isolated singular points corresponding to the apexes of these ordinary conifolds. These singular points lie on the fiber $\widetilde{\pi}^{-1}(\mathbf{o})$. Moreover, we will explicitly show that the exceptional fiber $\widetilde{\pi}^{-1}(\mathbf{o})$ consists of $(k-1)(l-1)$ copies of $\mathbb{P}^1 \times \mathbb{P}^1$. The map $\widehat{\pi}$ modifies only the singular points of $\widetilde{\mathcal{C}_{kl}}$ replacing each of them by a copy of the projective space $\mathbb{P}^1$. Thus $\widehat{\pi}$ is an isomorphism outside $(\widehat{\pi} \circ \widetilde{\pi})^{-1}(\mathbf{o})$. In particular, we have

$$\widehat{\pi}^{-1}(\widetilde{X_5}) \cong \widetilde{X_5}\,. \tag{2.18}$$

and $\widetilde{X_5}$ is smooth. As we mentioned above, $\widetilde{\mathcal{C}_{kl}}$ is smooth outside the kl ordinary conifold singular points. Thus if we put a large number of D3 branes at one of kl isolated singular points, denoted by $x_{ij}, 1 \le i \le k, 1 \le j \le l$, then the near-horizon limit of the geometry will be $AdS_5 \times \mathbb{T}^{1,1}$.

The 5 dimensional manifold $\widetilde{X_5}$ can be regarded as a smoothing of the singular Einstein manifold X_5. As we will see later, there will be $kl + k + l - 2$ number of 2-cyles and 3-cycles in $\widetilde{X_5}$ where kl is the number of the cycle coming from the horizon of each singular point of $\widetilde{\mathcal{C}_{kl}}$ and $k+l-2$ number of them comes by separating the above discussed two fixed circles in $X_5 = \mathbb{T}^{1,1}/\mathbb{Z}_k \times \mathbb{Z}_l$. As mentioned above, the first kind of these cycles corresponds to the 'diamonds' and the second kind corresponds to the 'stripes' in the brane box model. They correspond to stripes of boxes instead of individual boxes because we need to consider curves of either $\mathbf{A}_{k-1}$ or $\mathbf{A}_{l-1}$ singularity.

Before starting the actual discussion concerning the identification of the different 2-cycles, we present another proof for the fact that the Einstein manifold $\mathbb{T}^{1,1}$ is homeomorphic to $S^3 \times S^2$ [18, 19]. By changing coordinates

$$z_1 = w_1 + iw_2\,, \qquad z_2 = w_1 - iw_2\,, \qquad z_3 = w_3 + iw_4\,, \qquad z_4 = w_3 - iw_4\,, \quad (2.19)$$

we can rewrite the conifold equation (2.3) as:

$$w_1^2 + w_2^2 + w_3^2 + w_4^2 = 0\,. \quad (2.20)$$

Since the Einstein manifold $\mathbb{T}^{1,1}$ can be realized as a horizon (link) of the the conifold singularity, $\mathbb{T}^{1,1}$ is described by the intersection of (2.20) and the seven sphere in $\mathbb{C}^4$ given by

$$|w_1|^2 + |w_2|^2 + |w_3|^2 + |w_4|^2 = 1\,. \quad (2.21)$$

From (2.20) and (2.21), we see that $\mathbb{T}^{1,1}$ is given by

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 + x_4^2 = y_1^2 + y_2^2 + y_3^2 + y_4^2 &= \frac{1}{2}\,, \\ x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 &= 0\,, \end{aligned} \quad (2.22)$$

where x_i and y_i are the real and imaginary parts of w_i. Thus the y_i's describe a bundle of two spheres in the tangent bundle of S^3 given by the coordinates x_i's. Hence $\mathbb{T}^{1,1}$ is a sphere bundle S^2 over S^3. Since S^3 is parallelizable [47], a sphere bundle S^2 over S^3 is trivial and $\mathbb{T}^{1,1}$ is diffeomorphic to $S^2 \times S^3$. In fact, the frame for the sphere bundle over S^3 can be given by

$$\{(x_2, -x_1, -x_4, x_3), (x_3, x_4, -x_1, -x_2), (x_4, -x_3, x_2, -x_1)\}\,. \quad (2.23)$$

Next, we want to study the exceptional fiber $\widetilde{\pi}^{-1}(\mathbf{o})$. We will illustrate the general situation with $\widetilde{\mathcal{C}_{23}}$. To facilitate understanding, let us choose a basis for the

The vertices of the square	The coordinate rings The ideal of the fiber $\widetilde{\pi}^{-1}(\mathbf{o})$	The restriction of the map $\widetilde{\pi}$
$\{v_1, v_2, v_5, v_6\}$	$\mathbb{C}[z, y, xy^{-1}, xz^{-1}]$ (y, xz^{-1})	$a = z,\ b = y,\quad c = xy^{-1},\ d = xz^{-1}$ $p = b,\ q = b^2c^3,\ r = a^2d,\quad s = d$
$\{v_5, v_6, v_9, v_{10}\}$	$\mathbb{C}[z^{-1}, y, xy^{-1}, xz]$ (y, xz)	$a = z^{-1},\ b = y,\quad c = xy^{-1},\ d = xz$ $p = b,\quad q = b^2c^3,\ r = d,\quad s = a^2d$
$\{v_2, v_3, v_6, v_7\}$	$\mathbb{C}[z, x^2y^{-1}, x^{-1}y, xz^{-1}]$ (xz^{-1})	$a = z,\quad b = x^2y^{-1},\ c = x^{-1}y,\ d = xz^{-1}$ $p = bc^2,\ q = b^2c,\quad r = a^2d,\quad s = d$
$\{v_6, v_7, v_{10}, v_{11}\}$	$\mathbb{C}[z^{-1}, x^2y^{-1}, x^{-1}y, xz]$ (xz)	$a = z^{-1},\ b = x^2y^{-1},\ c = x^{-1}y,\ d = xz$ $p = bc^2,\ q = b^2c,\quad r = d,\quad s = a^2d$
$\{v_3, v_4, v_7, v_8\}$	$\mathbb{C}[z, x^3y^{-1}, x^{-2}y, xz^{-1}]$ (x^3y^{-1}, xz^{-1})	$a = z,\quad b = x^3y^{-1},\ c = x^{-2}y,\ d = xz^{-1}$ $p = b^2c^3,\ q = b,\quad r = a^2d,\quad s = d$
$\{v_7, v_8, v_{11}, v_{12}\}$	$\mathbb{C}[z^{-1}, x^3y^{-1}, x^{-2}y, xz]$ (x^3y^{-1}, xz)	$a = z^{-1},\ b = x^3y^{-1},\ c = x^{-2}y,\ d = xz$ $p = b^2c^3,\ q = b,\quad r = d,\quad s = a^2d$
$\{v_1, v_4, v_9, v_{12}\}$	$\mathbb{C}[y, x^3y^{-1}, xz, xz^{-1}]$	$(y, x^3y^{-1}, xz, xz^{-1})$

Table 1: The coordinate rings for the squares.

lattice $\mathbf{N}$ so that the coordinates of the lattice points are as follows:

$$\begin{array}{llll}
v_1 = (1,0,1)\,, & v_2 = (1,1,1)\,, & v_3 = (1,2,1)\,, & v_4 = (1,3,1)\,, \\
v_5 = (1,0,0)\,, & v_6 = (1,1,0)\,, & v_7 = (1,2,0)\,, & v_8 = (1,3,0)\,, \\
v_9 = (1,0,-1)\,, & v_{10} = (1,1,-1)\,, & v_{11} = (1,2,-1)\,, & v_{12} = (1,3,-1)\,.
\end{array}$$

The figure 1 shows the coordinate rings corresponding to the various squares of the toric diagram of $\widetilde{\mathcal{C}_{23}}$.

By tedious but direct computations from the figure 1, one can see that the fiber $\widetilde{\pi}^{-1}(\mathbf{o})$ consists of a union of $(k-1)(l-1)$ numbers of $\mathbb{P}^1 \times \mathbb{P}^1$. Moreover the adjacent components of the fiber meet along $\mathbb{P}^1$. If we denote each $\mathbb{P}^1 \times \mathbb{P}^1$ by a vertex and we join two of vertices if they meet, then we get a lattice of size $(k-1)$ by $(l-1)$. The figure 6 shows what they look like for $\widetilde{\mathcal{C}_{23}}$.

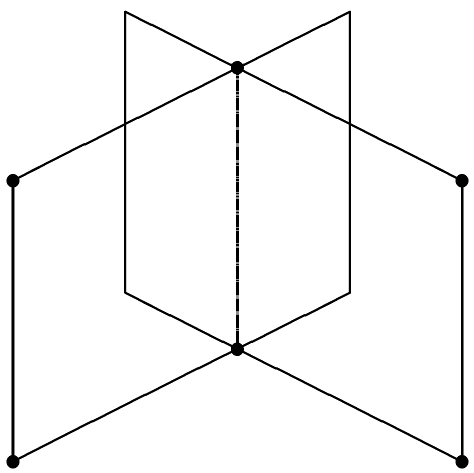

Figure 6: The fiber $\widetilde{\pi}^{-1}(\mathbf{o})$ for $\widetilde{\mathcal{C}_{23}}$.

In figure 6, each square represents $\mathbb{P}^1 \times \mathbb{P}^1$ and the singular points of $\widetilde{\mathcal{C}_{23}}$ are denoted by black dots. The fiber consists of two $\mathbb{P}^1 \times \mathbb{P}^1$ and they meet along $\mathbb{P}^1$. From this picture, one can see that the second betti number $h_2(\widetilde{\pi}^{-1}(\mathbf{o})) = k+l-2$. The D5 brane wrapping one of these $k+l-2$ spheres corresponds the fractional D3 brane coming from the

$\mathbb{Z}_k \times \mathbb{Z}_l$ twisted sector of the type-IIB supergravity on $AdS_5 \times \mathbb{T}^{1,1}/\mathbb{Z}_k \times \mathbb{Z}_l$. In the brane box model, this type of a fractional D3 brane turns into a D5 brane living on a stripe (see figure 4).

As we mentioned above, a full resolution $\widehat{\mathcal{C}_{kl}}$ of $\mathcal{C}_{kl}$ can be obtained by replacing the singular points of $\widetilde{\mathcal{C}_{kl}}$ by copies of $\mathbb{P}^1$. Hence the exceptional fiber $(\widehat{\pi} \circ \widetilde{\pi})^{-1}(\mathbf{o})$ will acquire kl copies of $\mathbb{P}^1$. This can be achieved by blowing up kl points on $\widetilde{\pi}^{-1}(\mathbf{o})$. Thus the second betti number of the exceptional fiber $(\widehat{\pi} \circ \widetilde{\pi})^{-1}(\mathbf{o})$ will be $k+l-2+kl$.

To study cycles on the Einstein manifold $\widetilde{X_5}$, consider an inclusion map

$$i : \widetilde{X_5} \hookrightarrow \widehat{\mathcal{C}_{kl}} \,. \tag{2.24}$$

First note that $\widehat{\mathcal{C}_{kl}}$ contacts to the exceptional fiber $(\widehat{\pi} \circ \widetilde{\pi})^{-1}(\mathbf{o})$, which we will denote by E. This contraction can be constructed by lifting a conical structure of $\mathcal{C}_{kl}$. We now regard $\widetilde{X_5}$ as a boundary of a smooth 6 dimensional manifold $\widehat{\mathcal{C}_{kl}}$. Consider a long sequence of homology groups with $\mathbb{Q}$ coefficients:

$$H_4(\widehat{\mathcal{C}_{kl}}, \widetilde{X_5}) \xrightarrow{\partial} H_3(\widetilde{X_5}) \xrightarrow{i_*} H_3(\widehat{\mathcal{C}_{kl}}) \xrightarrow{j_*} H_3(\widehat{\mathcal{C}_{kl}}, \widetilde{X_5}) \xrightarrow{\partial} H_2(\widetilde{X_5}) \xrightarrow{i_*} H_2(\widehat{\mathcal{C}_{kl}}) \,, \tag{2.25}$$

where j_* is induced by the inclusion $j : \widehat{\mathcal{C}_{kl}} \subset (\widehat{\mathcal{C}_{kl}}, \widetilde{X_5})$. Via the universal coefficient theorem and the Poincaré duality, we have

$$H_3(\widehat{\mathcal{C}_{kl}}, \widetilde{X_5}) \cong H^3(\widehat{\mathcal{C}_{kl}}) \cong H_3(\widehat{\mathcal{C}_{kl}}) = 0 \tag{2.26}$$

since $\widehat{\mathcal{C}_{kl}}$ can be deformed to E. Hence in the following commutative diagram, the top horizontal arrow will be injective and the bottom horizontal arrow will be surjective.

$$\begin{array}{ccc}
H_2(\widetilde{X_5}, \mathbb{Q}) & \xrightarrow{\quad i_* \quad} & H_2(\widehat{\mathcal{C}_{kl}}, \mathbb{Q}) \\
\wr\downarrow & & \wr\downarrow \\
H^3(\widetilde{X_5}, \mathbb{Q}) & & H^2(\widehat{\mathcal{C}_{kl}}, \mathbb{Q}) \\
\wr\downarrow & & \wr\downarrow \\
H_3(\widetilde{X_5}, \mathbb{Q}) & \xleftarrow{\quad \partial \quad} & H_4(\widehat{\mathcal{C}_{kl}}, \widetilde{X_5}, \mathbb{Q})
\end{array}$$

Here the vertical arrows are isomorphisms because of the universal coefficient theorem and the Poincaré dualities. Therefore, we conclude that

$$H_2(\widetilde{X_5}, \mathbb{Q}) \cong H_2(\widehat{\mathcal{C}_{kl}}, \mathbb{Q}) \,. \tag{2.27}$$

Moreover, we want to see the origin of these two cycles. Let L_{ij} be the horizon of $\widetilde{\mathcal{C}_{kl}}$ at x_{ij}. Then L_{ij} is isomorphic to $\mathbb{T}^{1,1}$ and L_{ij} does not change under the resolution $\widehat{\pi} : \widehat{\mathcal{C}_{kl}} \to \widetilde{\mathcal{C}_{kl}}$. From (2.27), we see that there is a natural inclusion

$$H_2(L_{ij}) \hookrightarrow H_2(\widehat{\mathcal{C}_{kl}}) \cong H_2(\widetilde{X_5}) \,. \tag{2.28}$$

Thus we may regard the 2-cycle of $H_2(L_{ij})$ as a 2-cycle of $H_2(\widetilde{X_5})$. We denote this 2-cycle by C^2_{ij}. By Poincaré duality, we may also regard the 3-cycle of $H_3(L_{ij})$ as a 3-cycle of $H_3(\widetilde{X_5})$, which will be denoted by C^3_{ij}. Note that there are kl contributions of 2-cycles from $H_2(L_{ij})$.

Moreover on each open neighborhood of x_{ij} represented by a square, we can choose a basis for one-forms

$$\begin{aligned} e^{\psi}_{ij} &= \frac{1}{3}\left(d\psi + \cos\theta_1\phi_1 + \cos\theta_2\phi_2\right) \\ e^{\theta_1}_{ij} &= \frac{1}{\sqrt{6}} d\theta_1 \qquad e^{\phi_1}_{ij} = \frac{1}{\sqrt{6}}\sin\theta_1 d\phi_1 \\ e^{\theta_2}_{ij} &= \frac{1}{\sqrt{6}} d\theta_2 \qquad e^{\phi_2}_{ij} = \frac{1}{\sqrt{6}}\sin\theta_2 d\phi_2 \,, \end{aligned} \tag{2.29}$$

so that the harmonic representatives of the second and third cohomology groups can be written as

$$\begin{aligned} e^{\theta_1}_{ij}\wedge e^{\phi_1}_{ij} - e^{\theta_2}_{ij}\wedge e^{\phi_2}_{ij} &\in H^2(L_{ij}) \\ e^{\psi}_{ij}\wedge e^{\theta_1}_{ij}\wedge e^{\phi_1}_{ij} - e^{\psi}_{ij}\wedge e^{\theta_2}_{ij}\wedge e^{\phi_2}_{ij} &\in H^3(L_{ij})\,. \end{aligned} \tag{2.30}$$

On the other hand, from the inclusion of $\widetilde{X_5}$ into the partially resolved conifold $\widetilde{\mathcal{C}_{kl}}$, we obtain a map

$$H_2(\widetilde{X_5}) \to H_2(\widetilde{\mathcal{C}_{kl}})\,. \tag{2.31}$$

Since we obtain $\widetilde{\mathcal{C}_{kl}}$ from $\widehat{\mathcal{C}_{kl}}$ by collapsing the blown-up two spheres which is a smooth deformation of the spheres in L_{ij}, we have the following exact sequence:

$$0 \to \bigoplus_{i=1,\ldots,k,j=1,\ldots,l} H_2(L_{ij}) \to H_2(\widetilde{X_5}) \to H_2(\widetilde{\mathcal{C}_{kl}}) \to 0\,. \tag{2.32}$$

Therefore we have obtained a concrete description of the cycles of $\widetilde{X_5}$ in terms of kl cycles from $H_2(L_{ij})$ and the $k+l-2$ cycles $H_2(\widetilde{\mathcal{C}_{kl}})$. The cycles in $H_2(\widetilde{\mathcal{C}_{kl}})$ are separating the singular points of $\widetilde{\mathcal{C}_{kl}}$. hence the corresponding cycles in $H_2(\widetilde{X_5})$ will separate the two fixed circles of X_5. This generalizes the results from the conifold and allows us to study the fluxes of NS-NS and R-R two forms in order to obtain logarithmic RG flow in the next section.

3. Fractional branes and RG flows

In this section we are going to extensively use the mathematical results of the previous section and identify the fractional branes as small perturbations of the string background. This will allow us to study the interpolation between the background with or without fractional D3 branes. This description will be shown to reproduce the logarithmic flow of gauge couplings,being in complete agreement with results of field theory.

We begin with a brief review of [1] where the RG flow determined by the fractional D3 branes was considered. Their result is a particular example of our case for $k = l = 1$. The coupling constant of field theory can be written in terms of the two-form charges on the vanishing sphere of the singularity:

$$\tau = C_0 + i\frac{1}{g^2} = \int_{C^2} B^{RR} + i\int_{C^2} B^{NS}\,, \tag{3.1}$$

where B^{RR}, B^{NS} are the R-R and NS-NS 2-form potentials. At the conifold point the values of the B-fields are fixed and the coupling constant is $g^{-2} \sim e^{-\phi}/2$. By wrapping M D5 branes over the 2-cycle of $\mathbb{T}^{1,1}$ in addition to N regular D3 branes orthogonal to the conifold, the string background will contain M units of RR 3-form flux through the 3-cycle of $\mathbb{T}^{1,1}$:

$$\int_{C^3} H^{RR} = M\,. \tag{3.2}$$

The equations of motion imply that H^{NS} should be proportional to M and to a product of the closed 2-form on $\mathbb{T}^{1,1}$ and a one form $\frac{df}{dr}dr = df(r)$ which involved dr. Hence the two form potential B^{NS} will be:

$$B^{NS} = e^{\phi} f(r)\omega_2, \quad \text{where} \quad f(r) \sim M\log r\,, \tag{3.3}$$

where ω_2 is the closed form on $\mathbb{T}^{1,1}$. The R-R scalar $C_0 = 0$ and the dilaton are set to have constant values.

The fractional D3 branes, obtained by wrapping D5 branes on the 2-cycle of $\mathbb{T}^{1,1}$ represent domain walls in AdS_5 and are obtained by wrapping D5 branes on the 2-cycle of $\mathbb{T}^{1,1}$. The relation between the two coupling constants of the field theory on the D3 branes can be written in the presence of M D5 branes wrapped on the 2-cycle as

$$\frac{1}{g_1^2} - \frac{1}{g_2^2} \sim e^{-\phi}\left(M\log r - \frac{1}{2}\right) \sim e^{-\phi} M\log r\,, \tag{3.4}$$

where the last relation is true in the large M approximation. This agrees with the field theory logarithmic RG flow equation in a non-conformal theory. The conformality is broken by the presence of the fractional D3 branes.

We now proceed to the case of the orbifolded conifold. The field theory on the world-volume of the N coincident D3 branes probing the singularity $\mathcal{C}_{kl}$ has been obtained in [7]. It is an $\mathcal{N} = 1$ chiral supersymmetric gauge theory with the gauge group

$$\prod_{i=1}^{k}\prod_{j=1}^{l} \mathrm{SU}(N)_{i,j} \times \prod_{i=1}^{k}\prod_{j=1}^{l} \mathrm{SU}(N)'_{I,j} \tag{3.5}$$

and with matter fields

Field	Representation
$(A_1)_{i+1,j+1;i,j}$	$(\square_{i+1,j+1},\overline{\square}'_{i,j})$
$(A_2)_{i,j;i,j}$	$(\square_{i,j},\overline{\square}'_{i,j})$
$(B_1)_{i,j;i,j+1}$	$(\square'_{i,j},\overline{\square}_{i,j+1})$
$(B_2)_{i,j;i+1,j}$	$(\square'_{i,j},\overline{\square}_{i+1,j})$

Moreover there is a quartic superpotential

$$W \sim \sum \Big((A_1)_{i+1,j+1;i,j}(B_1)_{i,j;i,j+1}(A_2)_{i,j+1;i,j+1}(B_2)_{i,j+1;i+1,j+1} - \\ - (A_1)_{i+1,j+1;i,j}(B_1)_{i,j;i+1,j}(A_2)_{i+1,j;i+1,j}(B_2)_{i+1,j;i+1,j+1}\Big) . \qquad (3.6)$$

As explained in the section 2, by taking T-duality, we obtain the brane box configuration consisting of k NS branes and l NS' branes whose intersections are smoothen out by diamonds. The singular point of $\mathcal{C}_{kl}$ splits into kl ordinary conifold singularities x_{ij} on $\widetilde{\mathcal{C}_{kl}}$ under the resolution $\widetilde{\pi} : \widetilde{\mathcal{C}_{kl}} \to \mathcal{C}_{kl}$ as in equation (2.16). Hence if we put a large-N number of D3 branes at each singular point x_{ij}, the the near-horizon limit of the geometry of $\widetilde{\mathcal{C}_{kl}}$ at x_{ij} will be $AdS_5 \times \mathbb{T}^{1,1}$. If we wrap a D5 brane over the 2-cycle of $\widetilde{X_5}$ corresponding to the 2-cycle of L_{ij}, we obtain a fractional D3 brane which is a domain wall in AdS_5 since it lies in the orthogonal direction to the D3 branes placed on the singular point x_{ij}. When we put one fractional brane together with N regular D3 branes, we will change the (i, j)-th copy of the SU(N) gauge group and the gauge group will change to SU$(N+1) \times$ SU$(N)^{kl-1} \times$ SU$(N)'^{kl}$ on the other side of the domain wall. The evidence for this claim is similar to the one of [27] i.e. by studying the behavior of wrapped D3-branes on 3-cycles of L_{ij} when they cross domain walls.

Before going further, we need to make a crucial observation concerning the constraint imposed by the consistency of the field theory on the worldvolume of the D3 branes. Since our field theory is chiral and can have anomalies, it is important to be careful with the way we introduce the fractional D3 branes. From the geometrical discussion, it appears that there is no restriction on introducing fractional D3 branes i.e. on wrapping D5 branes on different 2-cycles of the horizon. In the brane box picture this would mean that there is no restriction on the number of D5 branes on different diamonds. If there is no integer D3 brane in the theory, we can introduce any number of fractional D3 branes which correspond to D5 branes in a specific diamond. In the presence of D3 branes orthogonal to the conifold (integer D3 branes), we cannot put fractional D3 branes in only one diamond. This is because if we put one D5 brane in the (i, j)-th diamond, the gauge groups in the (i, j)-th and $(i+1, j+1)$-th boxes have one supplementary anti-fundamental field and the gauge groups in the $(i, j-1)$-th and $(i-1, j)$-th boxes have one supplementary fundamental field, these four gauge theories becoming anomalous. See figure 8 where we represent the ij diamond and fields which are in the fundamental or anti-fundamental representations. This determines a specific way to introduce fractional branes. We need to have either k fractional D3 branes corresponding to D5 branes on a row of diamonds or l fractional D3 branes corresponding to D5 branes on a column of diamonds. Then all the gauge groups in different boxes have the same number of fundamental and anti-fundamental fields and are anomaly free. When we discussed about the twisted sector in section 2, we saw that they also correspond to D5 branes on rows or columns of boxes so the filling of boxes and diamonds is similar.

We can now proceed to obtain the main goal of this section i.e. to compare the β-function calculation in field theory living on the world-volume of the integer D3 branes with the solution of supergravity equations of motion in the presence of fractional D3 branes. To start with, we need to discuss more about the "dibaryon" operators and the domain walls in AdS_5. Because $\widetilde{X_5}$ contains kl copies of Einstein manifolds $\mathbb{T}^{1,1}$, we can wrap D3 branes over the 3-cycle of each $\mathbb{T}^{1,1}$ to obtain kl types of "dibaryons". Besides we have integer D3 branes and D5 branes wrapped on each of the kl blow-up 2-cycles at each of the kl singular points, the latter ones being the domain walls. For $M_{i_0 j_0}$ D5 branes wrapped around the (i_0, j_0)-th 2-cycle, the gauge group changes from $\mathrm{SU}(N)^{kl} \times \mathrm{SU}(N)'^{kl}$ to

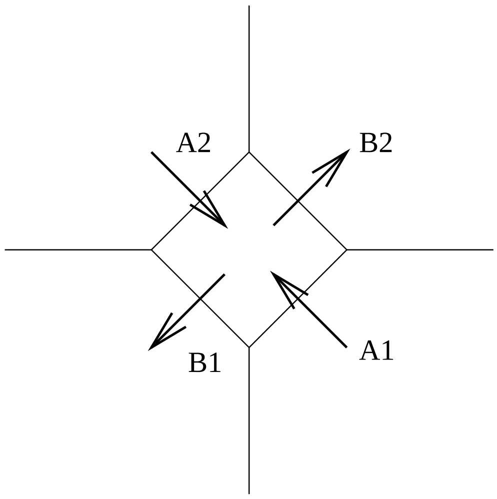

Figure 7: A diamond with the corresponding chiral fields.

$$\mathrm{SU}(N + M_{i_0 j_0}) \prod_{i=1}^{k} \prod_{j=1}^{l} \mathrm{SU}(N)_{ij} \times \mathrm{SU}(N)'^{kl}\,, \tag{3.7}$$

where the pair (i, j) does not take the value (i_0, j_0). As discussed before, field theory results require D5 branes on either rows or columns of diamonds. and this means that we need to have $M_{i_0 j}, j = 1, \ldots, l$ D5 branes wrapped around the $i_0, j = 1, \ldots, l$ cycles, the gauge group changing to

$$\prod_{j=1}^{l} \mathrm{SU}(N + M_{i_0 j}) \prod_{i=1, i\neq i_0}^{k} \prod_{j=1}^{l} \mathrm{SU}(N)_{ij} \times \mathrm{SU}(N)'^{kl}\,. \tag{3.8}$$

For $M_{i j_0}, i = 1, \ldots, k$ D5 branes wrapped around the $i = 1, \ldots, l; j_0$ cycles the gauge group changes to

$$\prod_{i=1}^{k} \mathrm{SU}(N + M_{i j_0}) \prod_{i}^{k} \prod_{j=1, j\neq j_0}^{l} \mathrm{SU}(N)_{ij} \times \mathrm{SU}(N)'^{kl}\,. \tag{3.9}$$

We can now proceed to construct the type-IIB dual to the $\mathcal{N}=1$ supersymmetric field theory with the gauge group (3.8) or (3.9). We have kl fluxes of RR 3-form through the 3-cycles C_{ij} which are Hodge duals to the 2-cycles surrounding the singular points x_{ij} for $i=1,\ldots,k, j=1,\ldots,l$:

$$\int_{C^3_{ij}} H^{RR} = M_{ij}\,, \qquad i=1,\ldots,k,\ j=1,\ldots,l\,. \tag{3.10}$$

Here we are identifying a 2-cycle of $H_2(L_{ij})$ with a 2-cycle of $H_2(\widetilde{X_5})$. To obey the above observed rule, one needs to turn fluxes through all $C^3_{i_0 j}, j=1,\ldots,l$ or all $C^3_{ij_0}, i=1,\ldots,k$ cycles with fluxes equal to $M_{i_0 j}$ or M_{ij_0} respectively, i_0, j_0 being some fixed indices.

We can now use the results of our previous section where we have completely identified the 2-cycles and the 3-cycles so the result is that the H^{RR} which we need to turn on are:

$$H^{RR} \sim \sum_j M_{i_0 j} e^{\psi}_{i_0 j} \wedge (e^{\theta_1}_{i_0 j} \wedge e^{\phi_1}_{i_0 j} - e^{\theta_2}_{i_0 j} \wedge e^{\phi_2}_{i_0 j})\,, \qquad \text{for fixed} \quad i_0 \quad \text{and} \quad j=1,\ldots,l \tag{3.11}$$

or

$$H^{RR} \sim \sum_i M_{ij_0} e^{\psi}_{ij_0} \wedge (e^{\theta_1}_{ij_0} \wedge e^{\phi_1}_{ij_0} - e^{\theta_2}_{ij_0} \wedge e^{\phi_2}_{ij_0}), \quad \text{for fixed} \quad j_0 \quad \text{and} \quad i=1,\ldots,k\,. \tag{3.12}$$

We now consider the type-IIB SUGRA equations of motion with the 2-form gauge potentials in the $AdS_5 \times X_5$ background with constant $\tau = C_0 + ie^{-\phi}$:

$$d * G = iF_5 \wedge G\,. \tag{3.13}$$

Here G is the complex 3-form field strength,

$$G = H^{RR} + \tau H^{NSNS}\,, \tag{3.14}$$

which satisfies the Bianchi identity $dG=0$.

If we choose $C_0 = 0$ and a constant dilaton, it follows from (3.11), (3.12) and (3.13) that

$$e^{-\phi} H^{NSNS} \sim \sum_j df_{i_0 j}(r) \wedge (e^{\theta_1}_{i_0 j} \wedge e^{\phi_1}_{i_0 j} - e^{\theta_2}_{i_0 j} \wedge e^{\phi_2}_{i_0 j})\,, \qquad \text{for fixed} \quad i_0 \tag{3.15}$$

or

$$e^{-\phi} H^{NSNS} \sim \sum_i df_{ij_0}(r) \wedge (e^{\theta_1}_{ij_0} \wedge e^{\phi_1}_{ij_0} - e^{\theta_2}_{ij_0} \wedge e^{\phi_2}_{ij_0})\,, \qquad \text{for fixed} \quad j_0 \tag{3.16}$$

are two solutions for the NS 3-form corresponding to the specific choice for H^{RR}. Since $F_5 = \mathrm{vol}(AdS_5) + \mathrm{vol}(X_5)$ and $d * H^{RR} = -e^{-\phi} F_5 \wedge H^{NSNS}$, we conclude

$$F_5 \wedge H^{NSNS} = 0 \tag{3.17}$$

and the real part of (3.13) is satisfied for all f_{ij}. From the imaginary part we have either

$$\frac{1}{r^3}\frac{d}{dr}\left(r^5\frac{d}{dr}f_{i_0j}(r)\right) \sim M_{i_0j}\,, \qquad \text{for fixed} \quad i_0 \tag{3.18}$$

or

$$\frac{1}{r^3}\frac{d}{dr}\left(r^5\frac{d}{dr}f_{ij_0}(r)\right) \sim M_{ij_0}\,, \qquad \text{for fixed} \quad j_0\,. \tag{3.19}$$

Thus we need to turn on an NS form as

$$B^{NSNS} \sim e^{\phi}\sum_j M_{i_0j}\omega_{i_0j}\log r, \text{for fixed} \quad i_0 \tag{3.20}$$

or

$$B^{NSNS} \sim e^{\phi}\sum_i M_{ij_0}\omega_{ij_0}\log r, \text{for fixed} \quad j_0\,. \tag{3.21}$$

The gauge couplings of the gauge theories are modified in the presence of the fluxes of the B^{NS} through the various 2-cycles. The gauge coupling without B-flux is related to the string coupling constant as $g^{-2} = \frac{1}{2g_s}$ where g_s is the string coupling constant. If all the diamonds and boxes have the same area, then field theories corresponding to D5 branes on boxes and diamonds have the same coupling constant and this the meaning of g in the previous formula. Since the B-fields (inverse of the gauge couplings) are areas on the torus, by changing the B-fluxes through the $(i_0, j), j = 1, \ldots, l$ or $(i, j_0), i = 1, \ldots, k$ cycles we change the areas of the diamonds. If we wrap D5 branes on the (i_0, j), $j = 1, \ldots, l$ or (i, j_0), $i = 1, \ldots, k$ cycles, the fluxes of B-field through the corresponding cycles modify and the gauge couplings change acoording to $\frac{1}{g_s}\int_{C^2_{i_0j}} B^{NSNS}$, $j = 1, \ldots, l$ or $\frac{1}{g_s}\int_{C^2_{ij_0}} B^{NSNS}$, $i = 1, \ldots, k$. The connection to the RG flow in field theory uses the relation:

$$\frac{1}{g_{i_0^2j}} - \frac{1}{g^2} \sim \frac{1}{g_s}\left(\int_{C^2_{i_0j}} B^{NSNS} - \frac{1}{2}\right), \qquad \text{for fixed} \quad i_0 \quad \text{and} \quad j = 1, \ldots, l \tag{3.22}$$

or

$$\frac{1}{g^2_{ij_0}} - \frac{1}{g^2} \sim \frac{1}{g_s}\left(\int_{C^2_{ij_0}} B^{NSNS} - \frac{1}{2}\right), \qquad \text{for fixed} \quad j_0 \quad \text{and} \quad i = 1, \ldots, k \tag{3.23}$$

The previous results for B^{NSNS} allow us to rewrite (3.22) as

$$\frac{1}{g^2_{i_0j}} - \frac{1}{g^2} \sim M_{i_0j}\log r\,, \qquad \text{for fixed} \quad i_0 \quad \text{and} \quad j = 1, \ldots, l \tag{3.24}$$

and (3.23) can be written as

$$\frac{1}{g^2_{ij_0}} - \frac{1}{g^2} \sim M_{ij_0}\log r\,, \qquad \text{for fixed} \quad j_0 \quad \text{and} \quad i = 1, \ldots, k\,. \tag{3.25}$$

Because the coordinate r is seen as a field theory scale in the AdS/CFT conjecture, relations (3.24) and (3.25) give the supergravity dual of the scale dependence of the difference between the gauge couplings.

The above results are obtained by solving the supergravity equations of motion and we are now going to compare them with β-function calculations in $\mathcal{N}=1$ supersymmetric field theory which give:

$$\begin{aligned} \frac{d}{d\log(\Lambda/\mu)}\frac{1}{g^2_{i_0 j}} &\sim 3(N+M_{i_0 j}) - 2N(1-\gamma_A-\gamma_B) \\ \frac{d}{d\log(\Lambda/\mu)}\frac{1}{g^2} &\sim 3N - 2(N+M_{i_0 j})(1-\gamma_A-\gamma_B)\,, \quad \text{for fixed } i_0 \text{ and } j=1,\ldots,l\,. \end{aligned} \tag{3.26}$$

For each diamond $(i_0 j)$ which belongs to the i_0-th row, the fields A and B which enter in the previous equations correspond to the bi-fundamental representations (A_1) in $(\Box_{i_0,j}, \overline{\Box}'_{i_0-1,j-1})$ of $\mathrm{SU}(N+M_{i_0 j})_{i_0,j} \times \mathrm{SU}(N)'_{i_0-1,j-1}$, (A_2) in $(\Box_{i_0,j}, \overline{\Box}'_{i_0,j})$ of $\mathrm{SU}(N+M_{i_0 j})_{i_0,j} \times \mathrm{SU}(N)'_{i_0,j}$, (B_1) in $(\Box'_{i_0-1,j}, \overline{\Box}_{i_0,j})$ of $\mathrm{SU}(N+M_{i_0 j})_{i_0,j} \times \mathrm{SU}(N)'_{i_0-1,j}$ and (B_2) in $(\Box'_{i_0,j-1}, \overline{\Box}_{i_0,j})$ of $\mathrm{SU}(N+M_{i_0 j})_{i_0,j} \times \mathrm{SU}(N)'_{i_0,j-1}$ where $\mathrm{SU}(N)'_{i,j}$ represents the gauge group on the (i,j) box and we use the fact that the $(i_0,j),(i_0,j-1),(i_0-1,j)$ and $(i_0-1,j-1)$ boxes are adjacent to the (i_0,j) diamond. The same for the (i,j_0) diamond, we obtain the formulas:

$$\begin{aligned} \frac{d}{d\log(\Lambda/\mu)}\frac{1}{g^2_{i j_0}} &\sim 3(N+M_{i j_0}) - 2N(1-\gamma_A-\gamma_B) \\ \frac{d}{d\log(\Lambda/\mu)}\frac{1}{g^2} &\sim 3N - 2(N+M_{i j_0})(1-\gamma_A-\gamma_B)\,, \end{aligned} \tag{3.27}$$

where γ are the anomalous dimensions of the fields A_1, A_2, B_1, B_2 and near the fixed point γ close to $-1/4$. By subtracting the second equation from the first in both (3.26) and (3.27), we obtain either

$$\frac{1}{g^2_{i_0 j}} - \frac{1}{g^2} \sim M_{i_0 j}[3+2(1-\gamma_A-\gamma_B)]\log\left(\frac{\Lambda}{\mu}\right) \tag{3.28}$$

or

$$\frac{1}{g^2_{i j_0}} - \frac{1}{g^2} \sim M_{i j_0}[3+2(1-\gamma_A-\gamma_B)]\log\left(\frac{\Lambda}{\mu}\right). \tag{3.29}$$

We use the identification of the spacetime radial coordinate r with the field theory scale and we see that the type-IIB supergravity solution has reproduced the field theoretic beta function, this establishing the gravity dual of the logarithmic RG flow in the $\mathcal{N}=1$ supersymmetric $\prod_{i=1}^{k}\prod_{j=1}^{l}\mathrm{SU}(N+M_{ij}) \times \mathrm{SU}(N)'^{kl}$ gauge theory on N regular and M_{ij}, $i=1,\ldots,k$, $j=1,\ldots,l$ fractional D3 branes. The agreement is between $1/g^2_{ij}N - 1/g^2 N$ at order M/N in the large-N limit.

In [48] the authors have obtained an analytic form for the gravitational RG flow in the gauged 5-d Supergravity in the case of $AdS_5 \times \mathbb{T}^{1,1}$. Their study concerned the back-reaction of the metric and 5-form fields. Their results could be generalized to our case with the difference that the local geometry around each of the kl singular points should be used instead of the global geometry.

In the case $l = 1$, the orbifolded $\mathbb{Z}_k \times \mathbb{Z}_l$ conifold becomes a generalized $\mathbb{Z}_k$ conifold. The horizon of the generalized conifold is singular and we need to partially resolve in order to obtain a smooth Einstein manifold horizon with k singular points. The procedure is just a particular case of our general recipe but the field theory on the worldvolume of D3 branes is not chiral in this case.

Acknowledgments

We would like to thank Keshav Dasgupta and Prabhakar Rao for stimulating discussions. The work of K. Oh is supported in part by NSF grant PHY-9970664.

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JHEP05(2000)030